On the Essence and Identity of Numbers*

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ABSTRACT: Taking as premises some reasonable principles about the essences of natural numbers, pluralities and sets, the paper offers two types of argument for the conclusions that the natural numbers could not be the Zermelo numbers, the von Neumann numbers, the "Kripke numbers", or the positions in the ω-structure, among other things. These conclusions are thus Benacerrafian in form, but it is emphasized that the two kinds of argument offered in the paper are anti-Benacerrafian in substance, as they are perfectly compatible and in fact congenial with some views on which the numbers could be things of certain other kinds.

Keywords: Arithmetic, Ontology, Benacerraf.

RESUMEN: Tomando como premisas algunos principios razonables sobre las esencias de los números naturales, las pluralidades y los conjuntos, el artículo ofrece dos tipos de argumentos para concluir que los números naturales no podrían ser ni los números de Zermelo, ni los números de von Neumann, ni los «números de Kripke», ni las posiciones en la estructura omega, entre otras cosas. Estas conclusiones son, por tanto, benacerrafianas en su forma, pero se enfatiza que los dos tipos de argumentos que se ofrecen en el artículo son anti-benacerrafianos en el fondo, puesto que son perfectamente compatibles con, y de hecho afines a, posiciones para las cuales los números podrían ser cosas de ciertos otros tipos.

Palabras clave: aritmética, ontología, Benacerraf.

1. *Two types of argument about what the numbers could not be*

The following is a reasonable principle about the natural numbers: provided the numbers are any things at all, a natural number must by its nature be susceptible of *being had* by pluralities of things, susceptible of being the number *of* pluralities. Thus, 17 is so susceptible, and in fact it is the number *of* the major moons of Jupiter—Callisto, Europa, Ganymede, Io, Amalthea, Himalia, Elara, Pasiphaë, Sinope, Carme, Lysithea, Ananke, Leda, Themisto, Adrastea, Metis, and Thebe: they *have* 17 as their number. It is in the nature of 17, and of numbers in general, to be completable in this way, the way of possession, by bunches of things. That the principle is reasonable could hardly be denied: our pretheoretical use of

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the notion of number evidently supports it, and there seems to be no reason to reject it in our philosophical theories of number. As it turns out, this principle suggests two types of argument, Benacerrafian as to the form of their conclusions but anti-Benacerrafian in substance, concluding that a variety of things, including Zermelo numbers, von Neumann numbers, "Kripke numbers" and the positions in the ω-structure,[1] could not be the natural numbers. These arguments are anti-Benacerrafian in substance because, unlike Benacerraf's argument, they are perfectly compatible and in fact congenial with some views on which the numbers could be things of certain other kinds.

Note first that the mentioned principle implies that in order for Zermelo 17, say, to *be* 17, Zermelo 17 must be susceptible of being had in the right way by pluralities of things, and presumably it must be such that it is had in the right way by Callisto, Europa, *et al.*, it must be such that these things *have* it in the appropriate sense. And surely Callisto, Europa, *et al.* bear, in some cases by necessity, a variety of relationships to Zermelo 17. For example, if *R* is the relationship in which some things and a thing stand when this thing is a set that results from as many iterations of the operation of unit set formation beginning with the empty set as there are things among those things, then Callisto, Europa, *et al.* bear (as a plurality) *R* to Zermelo 17.

Now it might be that one of the relationships of this kind were such that the property of bearing it to a given thing were the appropriate property some things have when they have that thing as their number; and if so, this relationship would also be such that the property of bearing its converse to some things would be the appropriate property a certain thing has when it is the number of those things. However, there are at least two reasonable constraints, one on properties such as the property of having 17 as a number, and another on properties such as 17's property of being had by pluralities of seventeen things, that reasonably exclude properties involving relationships of the mentioned kind to Zermelo numbers, von Neumann numbers, "Kripke numbers", positions in the ω-structure, etc.

The first constraint is this: a number *n* has essentially the property of being had by any plurality of *n* things. Note that the constraint is not that *n* is essentially in the *being had* relationship with particular pluralities of *n* things; clearly, for example, 17 is not had by Callisto, Europa *et al*. as a matter of what 17 essentially is. The constraint is rather that when we generalize over particular pluralities from properties such as the property of being had by a certain plurality of seventeen things such as the plurality of Callisto, Europa *et al.*, we obtain

[1] The Zermelo numbers are defined as follows: Zermelo 0 is ∅ (the empty set); and Zermelo n+1 is $\{n\}$. So the Zermelo numbers begin: ∅, {∅}, {{∅}}, ... The von Neumann numbers are defined as follows: von Neumann 0 is ∅; and von Neumann n+1 is $n \cup \{n\}$. So the von Neumann numbers begin: ∅, {∅}, {∅, {∅}}, ... A "Kripke number" is an ordered sequence of abstract objects corresponding to the digits in the number's representation in some base; Kripke has especially in mind the standard decimal Arabic representation, under which Kripke 543, say, would be the ordered sequence of the digits '5', '4' and '3' themselves, or the ordered sequence of the three set-cardinality properties of having five members, of having four members, and of having three members—or some related three-membered sequence. (Kripke 1992 proposes ideas of this general form as accounting for at least one use of (decimal Arabic) numerals. He nevertheless detects a kind of doublethink in our number talk, there being at least another use of numerals in which their denotations are thought of as being independent of idiosyncrasies of notation. On Kripke numbers see also Steiner 2011.) The notion of a position in the ω-structure will be explained below in the main text.

a purely general and essential property of the relevant number. For example, 17 essentially has the general property of being had by any plurality of seventeen things. And in general, a number *n* essentially has the general property of being had by any plurality of *n* things, it essentially has the purely general property that any plurality of *n* things will have it.

The second constraint is this: given *n* things, the property of having *n* as their number is part of the essence of those things taken as a plurality; any things having a number different from *n* as their number would be different (as a plurality) from those things, and not merely as a matter of what holds necessarily true of those things taken as a plurality, but in virtue of what we take the nature of the plurality of them to be.

In stating these constraints I am of course relying on the distinction between (merely) necessary and essential properties as elucidated by Kit Fine (1994, 1995). Fine noted that while all essential properties of a thing or things are necessary, the converse intuitively does not hold. In Fine's oft-repeated example, while it is a necessary property of Socrates to belong to his singleton, {Socrates}, it is intuitively not part of the essence or nature of Socrates to belong to his singleton. As Fine also puts it, belonging to his singleton is not a property that Socrates must have if he is to be the object that he is; by contrast, a truly essential property of Socrates, such as (we may suppose) being human, is a property Socrates must have if he is to be the object that he is. In some sense, Fine says, being human ought to appear in a specification of what Socrates is, of what being Socrates consists in, while belonging to {Socrates} is not part of such a specification. In this same sense, I am claiming, it is reasonable to adopt the constraints above. First, the purely general property that any plurality of *n* things will have it is part of the essence of a number *n*. This property ought to appear in a specification of what the number *n* is, for the property that any plurality of *n* things will have it is a property that the number *n* must have, if it is to be the thing that it is. And second, given *n* things, the property of having *n* as their number is part of the essence of those things taken as a plurality; this property ought to appear in a specification of what they are as a plurality, for having *n* as their number is a property that those things, taken as a plurality, must have, if they are to be the things that, as a plurality, they are.

Our two constraints independently imply, via reasonable steps, that the Zermelo numbers, the von Neumann numbers, the "Kripke numbers", the positions in the ω-structure, and many other things could not be the numbers. I will describe first the type of arguments given rise to by the first constraint; after that, I will turn to the type of arguments that arise from the second constraint.

17 has the property that any plurality of seventeen things will have it as its number. If this property of 17 is part of the essence of 17, as our first constraint states, then this property cannot be the property of bearing the converse of *R* to any plurality of seventeen things; for the property of bearing the converse of *R* to any plurality of seventeen things (which can only be had by Zermelo 17) is not part of the essence of Zermelo 17, even if it is probably a necessary property of Zermelo 17. It's not part of the essence of Zermelo 17 to be formed by as many iterations of the operation of set formation beginning with the empty set as there are things in pluralities of seventeen things. And it seems equally clear that no property consisting in the bearing of a relationship to any plurality of seventeen things will be part of the essence of Zermelo 17. Standing in such relationships ought not intuitively to appear in a specification of what it is to be Zermelo 17, it is not a characteristic that Zermelo 17 must have in order for it to be what it is. In fact, the notion of a plurality of seventeen things ought not intuitively to appear in a specification of what it is to be

Zermelo 17, it will not be involved in any characteristic that Zermelo 17 must have in order for it to be what it is. Zermelo 17 is simply (excuse the prolixity) the unit set of the unit set of the unit set of the unit set of the unit set of the unit set of the unit set of the unit set of the unit set of the unit set of the unit set of the unit set of the unit set of the unit set of the unit set of the unit set of the unit set of the unit set of the unit set of the empty set. Unlike 17, Zermelo 17 has its nature determined, so to speak, quite independently of its potential relationships to arbitrary pluralities of seventeen things.

The argument is then this: If no property consisting in the bearing of a relationship to arbitrary pluralities of seventeen things will be part of the essence of Zermelo 17, then, given that the property of bearing the relationship of being had to arbitrary pluralities of seventeen things is part of the essence of 17, Zermelo 17 is not (nor could it be) 17. And neither is n (nor could it be) its corresponding "Zermelo number", for any n.

(It is worth noting that the argument does not crucially depend on the supposition that numbers are numbers of pluralities, and can be run under the alternative (not necessarily incompatible) supposition that numbers are (also) numbers of sets. If numbers are numbers of sets, then a natural number will by its nature be susceptible of being had by sets of things, susceptible of being the number of sets. And certainly, as before, surely the set A = {Callisto, Europa, Ganymede, Io, Amalthea, Himalia, Elara, Pasiphaë, Sinope, Carme, Lysithea, Ananke, Leda, Themisto, Adrastea, Metis, Thebe} bears, in some cases by necessity, a variety of relationships to Zermelo 17 among which one might hope to find the appropriate *having* relationship. Take for example R' to be the relationship in which a thing and another thing stand when these things are sets and the latter thing is a set that results from as many iterations of the operation of unit set formation beginning with the empty set as there are elements in the first thing; then the set A bears the relationship R' to Zermelo 17. On the other hand, if numbers are numbers of sets, then the property of being had by any set of n things will intuitively be a part of the essence or nature of the number n. In other words, that property will be necessary to n, but also possessed by n in virtue of what n itself is. But then this property cannot be the bearing of the converse of R' to arbitrary sets of n things; for the bearing of the converse of R' to arbitrary sets of n things is not part of the essence of Zermelo 17, even if it is probably a necessary property of Zermelo 17. And it seems equally clear that no property consisting in the bearing of a relationship to arbitrary sets of n things will be part of the essence of Zermelo 17. Therefore, given that the property of being had by arbitrary sets of n things is part of the essence of 17, Zermelo 17 is not (nor could it be) 17. And neither is n (nor could it be) its corresponding "Zermelo number", for any n.)

Arguments of the same general form can be run for conclusions that the number 17 is none of a variety of other things that might be or have in fact been identified with the number 17. The corresponding argument that von Neumann 17 is not 17 is straightforward. And equally straightforward is the corresponding argument for the conclusion that "Kripke 17"[2] is not the number 17: it is enough to reflect that no property consisting in the

[2] Recall from the preceding note that Kripke 17 is an ordered sequence of two abstract objects corresponding to the digits "1" and "7", perhaps the ordered sequence of those two digits, or perhaps the ordered sequence of the two set-cardinality properties of having one member and of having seven members.

bearing of a relationship to arbitrary pluralities (or sets) of seventeen things will be part of the essence of the sequence <'1', '7'> or related sequences.

Only somewhat less straightforward is the corresponding argument for the conclusion that the number 17 is not the seventeenth position in the ω-structure.[3] Think of the ω-structure as a property satisfied by a system of objects just in case it is a progression;[4] and think of the seventeenth position in the ω-structure as the relation that obtains between a thing and a system of objects when the system is a progression and the thing occupies the seventeenth place in the system.[5] I claim that no property consisting in the bearing of a relationship to arbitrary pluralities (or sets) of seventeen things will be part of the essence of the seventeenth position in the ω-structure. For standing in such relationships does not intuitively appear in a specification of what it is to be the seventeenth position in the ω-structure, it is not part of the properties the seventeenth position in the ω-structure must have in order to be what it is. Thus, for example, it's not part of the essence of the seventeenth position in the ω-structure that it is such that the things in an arbitrary plurality of seventeen things could form the first seventeen elements in a progression such that it (the seventeenth position in the ω-structure) will apply to the pair formed by the seventeenth thing in the progression and the progression itself. In this, the notion of an arbitrary plurality of seventeen things is unlike the notions of an arbitrary system of objects, of a progression, and of occupying the seventeenth place in a progression, which do intuitively appear in a specification of what it is to be the seventeenth position in the ω-structure, as understood above. The seventeenth position in the ω-structure, unlike 17, has its nature determined independently of its potential relationships to arbitrary pluralities of seventeen things. For analogous reasons, neither is *n* (nor could it be) its corresponding "*n*th position in the ω-structure", for any *n*.

Let's now turn to the type of arguments that arise from the second constraint. Callisto, Europa, *et al.* have (as a plurality) 17 as their number. If this property of having 17 as their number is part of the essence of them as a plurality, as our second constraint has it, then this property cannot be the bearing of *R* to Zermelo 17; for the bearing of *R* to Zermelo 17 is not part of the essence of Callisto, Europa, *et al.* (as a plurality), even if it is probably a necessary property of them as a plurality. And it seems equally clear that no property consisting in the bearing of a relationship to Zermelo 17 will be part of the essence of Callisto, Europa, *et al.* (as a plurality). This is fairly clear at an intuitive level, but it is also reasonably supported by indirect considerations. Just as singleton Socrates is really an object extraneous to Socrates, one that is somehow built out of Socrates via the application of a certain procedure (set formation) that has nothing to do with the essence of Socrates, Zermelo 17 is an object extraneous to Callisto, Europa, *et al*. (as a plurality). It is therefore no wonder

[3] If we take 0 as a natural number, the thesis to be refuted will be strictly speaking that 17 is the *eighteenth* position in the ω-structure. But proponents of the thesis typically speak in the other, catchier way. If we like, we may suppose that there is a 0^{th} position in the ω-structure.

[4] By a progression it is meant, as usual, a system of objects together with a discrete linear order on them that has a first element and no last element, and such that the set of elements less than a given element is always finite.

[5] Stewart Shapiro (1997), ch. 3, proposes essentially to identify the numbers with the positions in the ω-structure.

that there is an intuition that a property involving Zermelo 17 cannot be part of the essence of Callisto, Europa, *et al.*

The argument against identifying 17 with Zermelo 17 that appeals to the second constraint is then this: If no property consisting in the bearing of a relationship to Zermelo 17 will be part of the essence of Callisto, Europa, *et al.* (as a plurality), then, given that the property of having 17 as their number is part of the essence of Callisto, Europa, *et al.* (as a plurality), Zermelo 17 is not (nor could it be) 17. And neither is *n* (nor could it be) its corresponding "Zermelo number", for any *n*.

(Again, the argument can be run under the supposition that numbers are numbers of sets. For any set *S* with *n* elements, the property of having *n* as its number is intuitively a part of the essence or nature of *S*. In other words, that property is necessary to *S,* but also possessed by *S* in virtue of what *S* itself is. Now *A* has 17 as its number. If this property of having 17 as its number is part of the essence of *A*, then this property cannot be the bearing of *R'* to Zermelo 17; for the bearing of *R'* to Zermelo 17 is not part of the essence of *A*, even if it is probably a necessary property of *A*. But it seems equally clear that no property consisting in the bearing of a relationship to Zermelo 17 will be part of the essence of *A*. Therefore, given that the property of having 17 as its number is part of the essence of *A*, Zermelo 17 is not (nor could it be) 17. And neither is *n* (nor could it be) its corresponding "Zermelo number", for any *n*.)

Conclusions that the number 17 cannot be identified with a variety of other things can be reached via arguments of the same general form. Again the corresponding argument that von Neumann 17 is not 17 is straightforward. As for the argument that "Kripke 17" is not the number 17, it is enough to reflect that no property consisting in the bearing of a relationship to the sequence <'1', '7'> or related sequences is part of the essence of Callisto, Europa, *et al.* as a plurality (nor part of the essence of *A*).[6]

The conclusion that the number 17 is not the seventeenth position in the ω-structure is again inevitable if one follows an analogous, if slightly more involved reasoning. The central claim is that no property consisting in the bearing of a relationship to the seventeenth position in the ω-structure is part of the essence of Callisto, Europa, *et al.* as a plurality (nor part of the essence of *A*). The key observation supporting this claim is that Callisto, Europa, *et al.* (and *A*) don't have any aspect of their essence as a plurality constituted by properties that involve the ω-structure, the property of being a progression. Surely the set of Callisto, Europa, *et al.* (*A*, that is) occupies the seventeenth position in some progressions, and so do each of its elements; also, Callisto, Europa, *et al.* simultaneously occupy the first seventeen positions in some progressions, and one can even construct some particular progressions in which they simultaneously occupy the first seventeen positions. But intuitively these or similar facts relating Callisto, Europa, *et al.* to the property of being a progression are not constitutive of any aspect of the essence of Callisto, Europa, *et al.* as a plurality (or of the essence of *A*), any more than the fact that Socrates is a member of some sequences and sets (let alone the fact that he is a member of many particular sequences and sets) is

[6] *A* may well have an essential property consisting in the bearing of a certain relationship to the cardinality property of having one member; for example, it may be of the essence of *A* to have subsets having one member. But I claim that no property consisting in the bearing of a relationship to the *ordered sequence* of the cardinality properties of having one member and of having seven members is of the essence of *A*.

constitutive of the essence of Socrates. To repeat, Callisto, Europa, *et al.* (and *A*) don't have an essential property consisting in the bearing of a relationship to the seventeenth position in the ω-structure.

2. The arguments under a sharper focus

The arguments of both the first and the second type rely on two premises each, one common to all arguments of the type and one specific to the kind of things that a particular argument concludes to be different from the numbers. In the case of the arguments of the first type, the common premise is that a number *n*'s property of being had by arbitrary pluralities of *n* things is a part of the essence of *n*. The specific premise in the case of the Zermelo numbers is the thesis that Zermelo *n* does not have any essential property involving a relationship to arbitrary pluralities of *n* things. In the case of the arguments of the second type, the common premise is that a plurality's property of having a certain number *n* is part of the essence of the plurality. The specific premise in the case of the Zermelo numbers is the thesis that a Zermelo number does not enter into the constitution of any essential property of a typical plurality. (*Mutatis mutandis*, here and in what follows, if we prefer a version of the argument in which it is sets that have numbers.) Both the common premise and some or all the specific premises in both types of arguments might be rejected on the grounds that the intuitions they are based on are not shared. This is undoubtedley a weakness of our types of arguments. A strength, however, is that the intuitions at stake arise within a conceptual framework—the framework of the distinction between essential and merely necessary properties as elucidated by Fine—whose intuitiveness has been well tested over time. Although Fine himself acknowledged the possibility that some readers might be blind to the distinction he was making, he stressed its intelligibility and trusted that many readers would find it compelling, an expectation that has evidently been borne out.[7] Rejecting the common premises or the specific premises we have employed because one does not see a distinction between essential and merely necessary properties will, I hope, be a correspondingly rare reaction among readers of the present arguments.

But if we take for granted the intelligibility of Fine's distinction, it is also hard to see how either the common premises or the specific premises that we have appealed to could be rejected. In the case of the specific premises of the second type of argument, for example, it is hard to see how a property involving Zermelo 17, even a necessary property of a typical plurality of seventeen members, such as the one above, could be held to fall on the essential side of the distinction. For Zermelo 17 would clearly seem to have nothing to do with the nature of a typical plurality of seventeen things such as the plurality of Callisto, Europa, *et al*. It would seem to have no role in a specification of what those things are (as a plurality), no role of involvement in the properties those things must have (as a plurality) if they are to be the things they are. As for the common premise of the first type of arguments, it does seem difficult to deny that it is of the essence of the number 17 to be had by arbitrary plu-

[7] Compare Sonia Roca-Royes' assessment on p. 66 of her (2011) survey, according to which Fine's examples establishing the relevant distinction have been accepted "to a degree that is unusual for philosophical arguments".

ralities of seventeen things; what else could the nature of 17 consist in, if it does not consist at least in this?

In the case of the specific premises of the arguments of the first type, it is hard to see, for example, how Zermelo 17 could have any essential property involving a relationship to arbitrary pluralities of *n* things, for Zermelo 17 would clearly seem to be constituted independently of any relationship to the notion of an arbitrary plurality of seventeen things. Perhaps it could be claimed that the notion of a plurality of seventeen things is somehow implicit in the characterization of Zermelo 17 as (forgive again the prolixity) the unit set of the empty set. In some sense it must be implicit in this that Zermelo 17 is the set that is obtained by applying, beginning with the empty set, the operation of set formation as many times as there are things in arbitrary pluralities of seventeen things. If one has enough extra information involving the notion of a plurality of things, one can derive this latter characterization from the first characterization, that specifies what it is to be Zermelo 17. But in order to establish a distinction between 17 and Zermelo 17 it would then be enough to point out that the notion of an arbitrary plurality of seventeen things is at any rate not related in the intuitively right way to Zermelo 17, if it only appears in the latter, "implicit" characterization of it. For 17 would appear to be in no merely implicit sense related to arbitrary pluralities of seventeen things, while Zermelo 17 would be only merely implicitly related, in a way that could only be extracted from its essence on the basis of extra information involving the notion of a plurality of things. This would already plausibly imply that 17 is not Zermelo 17.

In the case of the common premise of the arguments of the second type, its rejection leads (always assuming the intelligibility of Fine's distinction) to the claim that when some things have a certain number, their having this number is accidental to them as a plurality, so that having the number they have is not a property they must possess if they are to be the things they are. But how could the things in question be the things they are (as a plurality) if not, at least in part, because they have the number they have? Now one way of arguing that having the number they have is a property ultimately irrelevant to the essence of some things (as a plurality) would be as follows. Having the number 17 is a property that Callisto, Europa, *et al*. have in virtue of standing in a relation to a thing that, we might suppose, is after all *extraneous* to them—the number 17; but a property involving a thing extraneous to some things cannot be an essential property of these things (as a plurality); *ergo*, having the number 17 is not an essential property of Callisto, Europa, *et al*.

I think this way of arguing is unsuccessful. The important question is, of course, what reasons there are to think that the number 17 is extraneous to Callisto, Europa *et al.*, that it has nothing to do with what they are as a plurality, and how convincing these reasons are, as compared to the reasons why Zermelo 17 (and von Neumann 17, "Kripke 17", etc.) is extraneous to them. Surely it cannot be taken for granted that, in general, no property essential to a thing or some things involves things distinct from that thing or those things; the complicated debates about whether the origin of a person in a certain sperm and egg is essential to the person, and about whether a physical object has as a part of its essence the property of coming from a certain hunk of matter not too different from the hunk of

matter it actually came from, would get too easy a resolution if we took that for granted. But then what reason specific to the case of numbers can there be? It may be claimed, perhaps, that there is an intuition that 17 is just extraneous to Callisto, Europa *et al.*, period. But if this is all we have, the reasons for thinking that the number 17 is extraneous to Callisto, Europa *et al.*, do not even begin to compare to the reasons we have for thinking that Zermelo 17 (and von Neumann 17, "Kripke 17", etc.) is extraneous to them. The reasons for thinking that Zermelo 17 (and von Neumann 17, "Kripke 17", etc.) is extraneous to Callisto, Europa *et al.* are as conclusive as one might hope (provided Fine's distinction is taken to be intelligible): Zermelo 17 has uncontroversially nothing to do with the nature of Callisto, Europa *et al.* as a plurality. By contrast, if the only reason one might give for thinking that the number 17 is extraneous to Callisto, Europa *et al.* is that the number 17 is an object distinct from Callisto, Europa *et al.* (as a plurality), one would only have clearly insufficient reason for thinking that.

Furthermore, anyone having the intuition that 17 is just extraneous to Callisto, Europa *et al.*, period, will find it hard to accommodate it with the widely shared intuition that if Callisto, Europa, *et al.* are seventeen, *eo ipso* their number is 17. There is, so to speak, no further substantive metaphysical fact that has to obtain, once Callisto, Europa, *et al.* are seventeen, in order for them to have 17 as their number. This intuitively plausible principle is in turn underwritten by an evident feature of the pretheoretical use of the number notions: there is little or nil conceptual cognitive distance between the proposition that Callisto, Europa, *et al.* are seventeen, and the proposition that the number of Callisto, Europa, *et al.* is 17—competent speakers go from one to the other claim without experimenting any anxieties. And if it is in fact the case that when some things are *n* *eo ipso* their number is *n*, then, under the very plausible assumption that being seventeen is part of the essence of Callisto, Europa *et al.*, having 17 as their number is also part of their essence (as a plurality). The conditional intuition that if Callisto, Europa, *et al.* are seventeen *eo ipso* their number is 17, is more congenial, in view of the intuition that being seventeen is essential to Callisto, Europa, *et al.* (as a plurality), with a supposition that the number 17 must be a thing that in the relevant sense is not extraneous to Callisto, Europa, *et al.*; to suppose otherwise, in view of the conditional intuition, would certainly be to beg the question in favor of a specific conception of the numbers as extraneous to the pluralities that have them.

It might perhaps be claimed that the intuition that if Callisto, Europa, *et al.* are seventeen, *eo ipso* their number is 17, is more congenial with, or better explained by, the supposition that to say that Callisto, Europa, *et al.* have 17 as their number does nothing more, despite grammatical appearances, than to say that Callisto, Europa, *et al.* are seventeen. But this supposition will only help in the present case if it is further supposed that to say that Callisto, Europa, *et al.* are seventeen is not already to say that Callisto, Europa, *et al.* stand in a certain relationship to a certain thing that is their number. This is in itself a contentious supposition. And it is especially contentious in view of the possibility that a number is a property that somehow "inheres" in Callisto, Europa, *et al.*—a possibility that will take a specific, plausible shape in the next section, when we consider the possibility of identifying the numbers with cardinality properties.

It might then be claimed that a property of a plurality (or set) must be a thing extraneous to it, and thus that it cannot be involved in the constitution of its essential properties. But it is thoroughly unclear that all the properties of a thing or of some things can be extra-

neous to the thing or things in the appropriate sense.[8] It is unclear, in particular, that essential properties can be extraneous to the things to which they are essential, for there is a reasonably clear sense in which such properties have metaphysical precedence over the things to which they are essential: supposing, *per impossibile*, that they did not exist implies that the things in question, such as they are, would not exist either. While, in a reasonably clear sense, to suppose that belonging to singleton Socrates did not exist does not imply that Socrates, such as he is, would not exist, to suppose that being human did not exist does imply that Socrates, such as he is, would not really exist. In a similarly reasonably clear sense, to suppose that bearing *R* to Zermelo 17 did not exist does not imply that Callisto, Europa *et al.*, such as they are, would not exist (as a plurality). However, to suppose that having 17 as their number did not exist (or that 17 did not exist) does seem to imply that Callisto, Europa *et al.* would not exist such as they are (as a plurality); for if Callisto, Europa *et al.* (*per impossibile*) did not have 17 as their number, they would not be seventeen, and yet they are essentially seventeen.[9]

It thus seems natural to argue as follows in favor of the principle that having *n* as their number is essential to a plurality of *n* things. In a reasonably clear strong sense of "*eo ipso*", it is not the case that if Callisto, Europa *et al.* are seventeen, then *eo ipso* they bear *R* (say) to Zermelo 17; Zermelo 17 has nothing to do with Callisto, Europa *et al.*, so it is surely not the case that if Callisto, Europa, *et al.* are seventeen *eo ipso* they bear *R* (say) to Zermelo 17. But it is reasonable to expect, at least if we assume that numbers exist, that if Callisto, Europa, *et al.* are seventeen *eo ipso* they have 17 as their number. And it is hard to see how this expectation could be fulfilled unless 17 were not extraneous to Callisto, Europa, *et al.* (as a plurality). For if we assume that if Callisto, Europa, *et al.* are seventeen *eo ipso* they have 17 as their number, and that Callisto, Europa, *et al.* (as a plurality) are essentially seventeen, then having 17 as their number must be an essential property of Callisto, Europa, *et al.* (as a plurality). And if 17 were extraneous to Callisto, Europa, *et al.* (as a plurality), then having 17 as their number would not be an essential property of Callisto, Europa, *et al.* (as a plurality). The reasonable course of action is thus to suppose that 17 is not extraneous to Callisto, Europa, *et al.* (as a plurality), and that having 17 as their number is essential to Callisto, Europa, *et al.* (as a plurality).

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[8] Fine (1995), 60, does adopt the view that properties, at least typically, are extraneous to a thing, even if they are essential properties of it. But, as he notes, this requires imposing a pretty special restriction on the "closure" idea that the properties logically implied by the essential properties of a thing ought to be also essential properties of it. If Socrates is essentially human, then by property abstraction, usually considered a logical rule, and the closure idea, Socrates essentially has the property of being human. On my own conception, property abstraction is indeed a logical rule, and there is no need to restrict the closure idea in this case, as the principle that the essential properties of a thing are not extraneous to it is perfectly acceptable and defensible.

[9] The fact that some properties seem to have metaphysical precedence over the things of which they are properties provides in itself one way of arguing that 17 is not Zermelo 17 (say). Having 17 as their number is at the very least a candidate for a property having metaphysical precedence over Callisto, Europa *et al.* But being in the relationship *R* (or similar relationships) to Zermelo 17 has in no sense metaphysical precedence over Callisto, Europa *et al.*

3. On what the numbers could be

The present types of argument have a feature that might at first sight be thought of as a limitation, as compared with Paul Benacerraf's (1965) classical argument. While the arguments of the present types provide alternative ways of reaching a variety of conclusions which are Benacerrafian in form, the arguments are by no means as sweeping as Benacerraf's classical argument. Benacerraf's argument was that since any thing could play the role assigned to 17 in a normal person's education insofar as any thing could be the seventeenth member in a progression, no thing has a stronger claim than any other to be identified with the number 17, and hence no thing in particular is to be identified with the number 17. The present types of argument, by contrast, rely on the idea that the role assigned to 17 in a normal person's education, insofar as this education inculcates or elicits in some way the intuition that numbers and pluralities (and/or sets) in particular have natures or essences of certain kinds, cannot be played by a variety of things, including Zermelo numbers, von Neumann numbers, "Kripke numbers", positions in the ω-structure, etc. The present types of argument are thus perfectly compatible with some views on which the numbers could be (or just are) things of certain kinds. For in relying on the idea that the role assigned to 17 in a normal person's education cannot be played by a variety of things, they presuppose that this education inculcates or elicits in some way some requirements on the number 17 that go beyond Benacerraf's only detected requirement, that it be susceptible of being the seventeenth element in a progression.

But I don't see this feature of the present types of argument as a limitation. Far from it, I see it as pointing towards some positive considerations on the issue of what the numbers are or could be. For the requirements the present types of argument appeal to, viewed as requirements on the numbers inculcated or elicited by a typical education, could have a role in determining what the numbers are or could be.

While I am by no means completely certain that the relevant requirements are enough to determine what thing the number 17 is, the common premises of the two types of argument developed in this paper are in syntony with existing views on which the numbers are cardinality universals of some sort. These views clearly satisfy, or at least have the potential for satisfying, both the requirement that the property of being had by arbitrary pluralities (or sets) of seventeen things is essential to 17 and the requirement that the property of having 17 as their number is part of the essence of Callisto, Europa *et al.* as a plurality (or the requirement that the property of having 17 as its number is part of the essence of *A*). For it is reasonable to accept that the bearing of *instantiation* to a certain cardinality universal is part of the essence of Callisto, Europa, *et al.* as a plurality (or that the bearing of related relations to cardinality universals is part of the essence of *A*), and that the bearing of the converse of instantiation to arbitrary pluralities of seventeen things is part of the essence of an appropriate cardinality universal. Take the view, close to my sympathies, that the numbers are plural attributive cardinality properties—not universals appropriately related to sets, but properties instantiated plurally by some things or others.[10] Since it is intuitively of the essence of Callisto, Europa *et al.*, considered plurally, to be seventeen, to instantiate the plural cardinality property of being seventeen, the property of having 17 as their number can

[10] See e.g. Byeong-Uk Yi (1999).

be taken to be part of the essence of Callisto, Europa *et al.* as a plurality if 17 is identified with the plural cardinality property of being seventeen and the relation of having at stake is understood as instantiation. Thus, such views can satisfy our second constraint above. Also, since it is intuitively of the essence of the plural cardinality property of being seventeen to be instantiated precisely by arbitrary pluralities of seventeen things, the property of being had by arbitrary pluralities of seventeen things can be taken to be part of the essence of 17, if 17 is identified with the plural cardinality property of being seventeen and the relation of being had at stake is understood as the converse of instantiation. Thus such views can satisfy our first constraint above.

(Or take views on which the numbers are either cardinality *kinds* of finite sets[11] or cardinality *attributes* of finite sets.[12] Kinds paradigmatically constitute aspects of the essences of things—it is by virtue of belonging to its kind that an important part of the essence of the thing is constituted, and sets, if they indeed have the numbers as cardinality kinds to which they belong, would be no exception. The property of having 17 as its number can be taken to be part of the essence of *A* if the relation of having at stake is understood as membership in a cardinality kind. Also, many attributes that are not kinds can be constitutive of aspects of the essence of a thing; and sets, if they indeed have the numbers as cardinality attributes that qualify them, would be no exception. Such views can clearly satisfy the requirement that the property of having 17 as its number is part of the essence of the set with Callisto, Europa *et al.* as elements, i.e. *A*. For it is intuitively of the essence of *A* to instantiate its set-cardinality property, and so the property of having 17 as its number can be taken to be part of the essence of *A* if the relation of having at stake is understood as instantiation of a set-cardinality property. On the other hand, the kind (/attribute) of having seventeen members has intuitively as a part of its essence the property of being instantiated by precisely the sets with seventeen members, so it again satisfies the constraint that the number 17 must have as a part of its nature the property of being had by arbitrary sets of seventeen things, if the relation of being had at stake is understood as the converse of instantiation.)

Benacerraf, following Richard M. Martin, claimed that arithmetic, as studied and developed by mathematicians, concerns itself only with structural properties of any progression. But he chastised Martin for viewing with approval the philosopher's dissatisfaction with the structuralist outlook apparently presupposed by mathematicians. If it is a fact that mathematicians presuppose a structuralist outlook, can't this fact be turned into an argument that, contrary to some of the intuitions we have been appealing to in this paper, there is actually no such thing as the nature of a number or the essence of some things as a plurality, or the essence of a set, or even no such things as the numbers or the sets? For it might be tempting to think that mathematicians must study all relevant aspects of numbers and sets, and so that if they do not study their alleged natures or essences, then these alleged natures or essences must not exist.

I will not deny that this thought may be tempting for some, but I am almost certain that it is ultimately incorrect. Even if it is really true that mathematicians concern themselves exclusively with structural properties of certain kinds, it is no less true that their original motivation for doing this in the cases of some mathematical theories lies in the idea that there is some particular system or realization that instantiates those structural properties

[11] See e.g. E. J. Lowe (1998), ch. 10.
[12] See e.g. Penelope Maddy (1981).

they seek to study and isolate. For example, this is arguably the case with Euclidean geometry and (idealized) physical space, or with classical analysis and the (idealized) physical real line. It is also arguably the case with set theory as modernly conceived and the (presumed) cumulative hierarchy with the empty set at its basis. To the extent that the empty set is thought of as an object having a particular nature or essence (and to the extent that the operation of set formation is thought of as a particular operation yielding a definite object when applied to a plurality of objects), so is the cumulative hierarchy (or at least so is any typical ontologically less controversial but still huge standard model of set theory) thought of as an object having a particular nature or essence, however inconceivably complex. Nevertheless, even though set theorists often speak as if they were talking about this object, in fact all their results would hold for any structure isomorphic to the cumulative hierarchy with the empty set at its basis, e.g. for an isomorphic structure with Julius Caesar at its basis, in which the relevant cumulative operation need not even be the operation of set formation, but some other extensional operation that yields a definite object when applied to a plurality of objects. Now I take it that this observation about what set theorists really care about does not mean that there is no such thing as the empty set or its essence. (There may be other reasons for thinking so, but not this.) Similarly, I do not think that Benacerraf's claim that theoretical arithmetic concerns itself only with structural properties, even if it is correct, can imply that there are no such things as the numbers or their essences. And if this is so, the way is open to thinking that, while there are many things that the numbers could not be, there are some things that, for all we know, the numbers could be.

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